PHYSICAL REVIEW E VOLUME 50, NUMBER 3 SEPTEMBER 1994

Nonlinear feedback for controlling the Lorenz equation

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(Received 20 April 1994)

This work presents a nonlinear feedback approach for controlling the Lorenz equation. The derivation of the feedback is based on linearizing an input-output dynamic of the system, which leads to large regions of asymptotic stability. Here the input signal to the Lorenz equation is the applied heat via the Rayleigh number. The performance of the nonlinear feedback is tested via the stabilization of equilibrium points and periodic orbits.

PACS number(s): 05.45.+b

I. INTRODUCTION

Controlling chaos has become a challenging topic in the field of nonlinear dynamics [1]. Although it is possible to modify the dynamics of a chaotic system by injecting time-dependent signals *(nonfeedback control)* [2], it is accepted that feedback control offers more advantages, such as robustness and signal tracking [1,3–5]. In this work we focus on the feedback control of a continuous time system: the Lorenz equation.

Roughly speaking, there are two ways of feedback controlling a chaotic system: (i) control based on a Poincaré map of the system and (ii) continuous control. This first approach was suggested by Ott, Grebogi, and Yorke (OGY) [1]. The main drawback of such an approach is that the system trajectory must reach a neighborhood of the desired trajectory (in most cases, a periodic orbit) on the Poincaré section in order to active the control action. This leads to large stabilization times and poor robustness properties. In principle, it is desirable to have the control activated all the time in order to induce stronger modifications of the dynamics. This idea leads to the use of *continuous controls* [6]. Continuous control is the typical approach in control theory [5,11].

The Lorenz equation is a well known chaotic system, which has been used for control studies [7–9]. Regarding the control of the Lorenz system, and following Hartley and Mossayebi's work [9], the control input was arbitrarily added with little consideration for actual physics. In particular, Qu, Hu, and Ma [8] included a manipulated additive term to the original equation. The main criticism of such an approach is that the derived feedback control is hardly physically realizable. From the applications viewpoint, control inputs must be related to actual physical parameters [9,10]. In a recent work, Hartley and Mossayebi [9] used the applied heat via the Rayleigh number as the control input to derive a continuous feedback control. Their control derivation is based on the Taylor linearization of the system dynamics in a reference point. Classical results in linear control theory [11] assure that stabilization of the linear system implies local stabilization of the nonlinear one. However, when the desired dynamics is a nontrivial periodic orbit, the problem arises of choosing an "adequate" Taylor linearization. In this work, we derive a nonlinear feedback control which is useful to stabilize both equilibrium points and periodic (or nonperiodic) trajectories.

The work is organized as follows. The next section presents the derivation of the feedback controller and states some closed-loop stability properties. After this analysis, the performance of the resulting feedback is evaluated via numerical simulations.

II. DERIVATION OF THE FEEDBACK CONTROL

The Lorenz system is given by [12]

$$\begin{aligned} \dot{x} &= P(x-y) = f_1(x,y,z) , \\ \dot{y} &= Rx - y - xz = f_2(x,y,z) , \\ \dot{z} &= xy - bz = f_3(x,y,z) , \end{aligned} \tag{1}$$

where $P>0$ is the Prandtl number, $b>0$ is a constant, and R is the Rayleigh number. R will be allowed to vary, so that R is taken as the control input. The system (1) is related to a fluid thermosiphon [9,10] in the following way: R is proportional to the heat applied to the bottom half of the fluid, x is the fluid velocity in the loop, y is the vertical temperature difference, and z is the horizontal temperature difference (see Fig. 1).

We need the following observation to derive the feedback control: if the vertical temperature difference y is controlled to a fixed value y_{ref} via the applied heat R, then $dy/dt=0$ and the system (1) is reduced to the next one:

$$\begin{aligned} \dot{x} &= -Px + Py_{\text{ref}} , \\ \dot{z} &= -y_{\text{ref}}x - bz , \end{aligned} \tag{2}$$

which is a linear system with eigenvalues $\{-P,-b\}$, so therefore the system (2) is asymptotically stable. Hence, in principle, it is sufficient to stabilize the vertical temperature difference y to stabilize the complete system (1). To achieve such an objective, we will follow a nonlinear approach. The central idea is to linearize an input–output (y) dynamic of the system (1). Formally, such objective can be achieved via the singular feedback

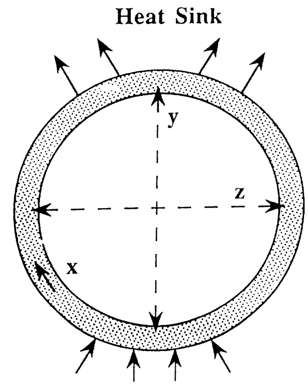

FIG. 1. Schematic description of a thermosiphon.

$$R(x,y,z)=(v+y+xz)/x \ , \tag{3}$$

where v is a new control input. Substituting (3) in (1) yields

$$\dot{x}=P(x-y) \ ,$$
$$\dot{y}=v \ , \tag{4}$$
$$\dot{z}=xy-bz \ ,$$

which is a linear system in cascade with a nonlinear one. In this way, v can be chosen in such a way that the system (4) is asymptotically stable [for instance, $v=-K(y-y_{\mathrm{ref}})$, $K>0$]. The main problem with the feedback (3) is its singularity at the set $\Sigma=\{(x,y,z):x=0\}$. At $\Sigma\subset\mathbb{R}^3$, the system (1) is not sensible to control actions [in other words, the Lorenz system (1) is not controllable at Σ, which corresponds to zero flow velocity into the thermosiphon]. When a physical trajectory approach Σ, the control feedback asks for infinite control signals. Furthermore, when a trajectory crosses Σ, $R(x,y,z)$ changes from $+\infty$ to $-\infty$ (or from $-\infty$ to $+\infty$), so that there is not a continuous and bounded function $\overline{R}(x,y,z)$ that arbitrarily approximates $R(x,y,z)$. From a physical viewpoint, the above control actions in the set Σ imply that one must make drastic (in fact, discontinuous) changes in the heating operation at the bottom of the loop. As in the case of resolution of singularities in algebraic curves, the singularity of the feedback control at the set Σ can be regularized (in some sense) by enlarging the dimension of the working space. Consider R as a function of time. We take the second time derivative of y along the vector field of (1),

$$\ddot{y}=Rf_1-f_2-xf_3-f_1z+x\dot{R} \ , \tag{5}$$

where we have used f_i for $f_i(x,y,z)$. If dR/dt is taken as

$$\dot{R}=(-Rf_1+f_2+xf_3+zf_1+v)/x \ , \tag{6}$$

the closed-loop dynamics becomes $d^2y/dt^2=v$, which is a linear differential relation between the control input v and the controlled variable y. Since v can be arbitrarily assigned, we set $v=k_1f_2+k_2(y-y_{\mathrm{ref}})$ to stabilize an equilibrium point of (1) related to y_{ref}. If the objective is to track a nontrivial reference signal $y_{\mathrm{ref}}(t)$ (maybe a periodic orbit), the input v must be chosen as

$$v=d^2y_{\mathrm{ref}}/dt^2+k_1(f_2-dy_{\mathrm{ref}}/dt)+k_2(y-y_{\mathrm{ref}}) \ .$$

Here k_1 and k_2 are chosen in such a way that the roots of the polynomial $P(r)=r^2-k_1r-k_2$ lie in the left half of the complex plane. Observe that the control input R satisfies a differential equation, which must be integrated together with the Lorenz system (1). In the control theory literature [5], (6) is known as a *first-order dynamical compensator.* Assuming that y_{ref} is constant (equilibrium point), under the following singular change of coordinates:

$$\begin{aligned} z_1&=y-y_{\mathrm{ref}} \ ,\\ z_2&=Rx-y-xz \ ,\\ z_3&=x \ ,\\ z_4&=z \ , \end{aligned} \tag{7}$$

the controlled system (1), (6) becomes

$$\begin{aligned} \dot{z}_1&=z_2 \ ,\\ \dot{z}_2&=k_1z_1+k_2z_2 \ , \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{z}_3&=P(z_1-z_3) \ ,\\ \dot{z}_4&=z_3z_1-bz_4 \ , \end{aligned} \tag{8b}$$

which is globally, asymptotically stable at the origin. Analogous stability conclusions can be stated for the case of the signal tracking $y_{\mathrm{ref}}(t)$. Note that the singularity $x=0$ has been transferred from the control signal in (3) to the vector field that generates R in (6), which implies that a crossing of a trajectory of (1),(6) induces a blow-up in the time derivative of R. At a crossing, the control signal $R(t)$ can be arbitrarily approximated by a continuous one $\overline{R}(t)$ [i.e., $|R(t)-\overline{R}(t)|\le\delta$, with $\delta>0$ arbitrarily small]. In this way, the feedback (6) is physically realizable. The feedback (6) can be interpreted as a first-order filter that regulates the feedback (3).

Resuming, one can think of the closed-loop system (1),(6) as a piecewise asymptotically stable linear system with closed-loop trajectories $\phi_t(x_0,y_0,z_0,R_0)$ suffering instantaneous bursting when crossing the singularity set $\Sigma'=\Sigma\times\mathbb{R}$. Although the regularization of singularities is actually an open problem in control theory, we conjecture that higher-dimensional dynamical extensions (that is, R satisfies an nth-order differential equation) smooth the control signal $R(t)$ in a more efficient way. In this work, we restrict ourself to the first-order extension (6). The next sections present simulations with the controller (6).

III. STABILIZATION OF EQUILIBRIUM POINTS

As in most studies, in what follows we set $P=10$ and $b=\frac{8}{3}$. Also, we will take the stationary value of the control signal $R_{\mathrm{ref}}=28$ (the usual value for chaotic behavior). The system equilibrium points will first be found. Let capital letters designate equilibrium points. The first equation gives $X=Y$; the last two equations give

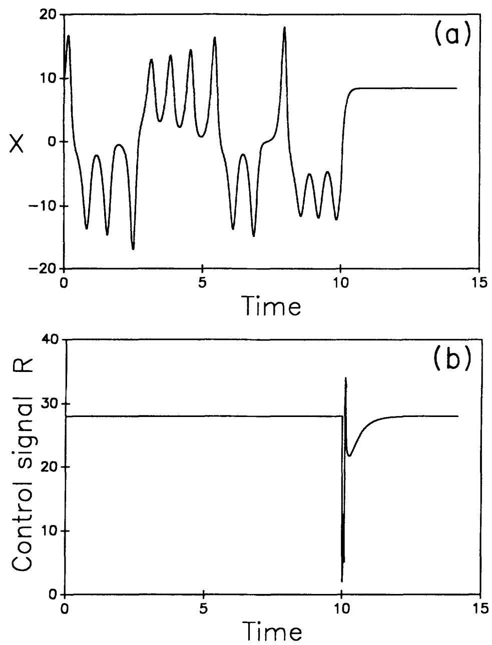

FIG. 2. (a) Velocity of the fluid in the thermosiphon. The control is activated at $t=10$ to stabilize the equilibrium point (8.485,8.485,27). (b) Control actions corresponding to the stabilization of the equilibrium point (8.485,8.485,27).

$$X=Y=\pm\sqrt{8(R_{\text{ref}}-1)/3}=\pm 8.485 ,$$

and $Z=R_{\text{ref}}-1=27$. In addition to these two equilibrium points, there is another one at $(X,Y,Z)=(0,0,0)$, which does not depend on the stationary value R_{ref} [i.e., the origin is an equilibrium point of (1) for all $R\in\mathbb{R}$]. Although this equilibrium is able to be stabilized, it has no physical significance because its existence is related to the absence of fluid flow in the thermosiphon loop. By symmetry, it is sufficient to consider stabilization at $(X,Y,Z)=(8.485,8.485,27)$. As a consequence, $y_{\text{ref}}=8.485$. By setting $k_1=-400$ and $k_2=-40$, the input (v)–output (y) linear system possesses the eigenvalues $\lambda_1=\lambda_2=-20$, which are used in all numerical simulations (the value -20 was chosen as the base for numerical simulations). In addition, the following bounds have been imposed: $0\le R(t)\le 50$. A simulation using the controller (6) is given in Figs. 2(a) and 2(b). The control is achieved at $t=10$ with $R(t=0)=28$. Observe that the stabilization of the equilibrium point is preceded by a short transient behavior.

IV. STABILIZATION OF PERIODIC ORBITS

In OGY's work, a feedback based on a Poincaré section was used to stabilize periodic orbits embedded in chaotic attractor [13]. However, in experiments (even in numerical simulations) it is difficult to find unstable periodic orbits. As a consequence of the denseness of the chaotic orbit, there is an uncountable number of nearly periodic orbits (NPO's) which can be used as an alternative to periodic orbits. The NPO's $\phi_t(x,y,z)$ are finite time T aperiodic pieces of a more complicated (chaotic or

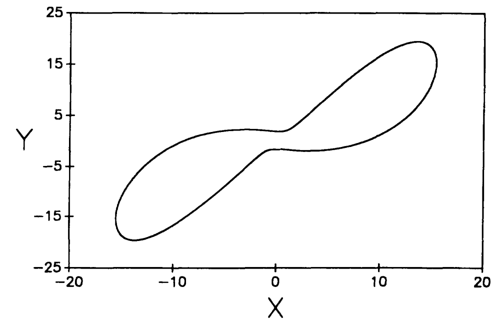

FIG. 3. Symmetric NPO of the Lorenz system (1).

nonchaotic) trajectory, of which the ending position $\phi_T(x,y,z)$ is very near to the starting point $\phi_0(x,y,z)$. A NPO has the following characteristics: (i) it is not a complete trajectory of the uncontrolled system (1); (ii) due to the sensitivity to initial conditions, it is inherently unstable. Therefore, without the action of a control, the system (1) cannot track a NPO. Figure 3 shows a symmetric NPO. Figure 4(a) shows the velocity in the loop as a function of time; the control is activated at $t=10$ as shown in Fig. 4(b). Note that, after a transient behavior, the velocity x follows a periodic orbit. On the other hand, after such transient behavior, the control signal $R(t)$ takes values in the neighborhood of $R_{\text{ref}}=28$, which corroborates the fact that the reference NPO is a piece of a complete trajectory of (1) with $R=R_{\text{ref}}$. Therefore, except for some isolated points, the NPO in Fig. 3 exists

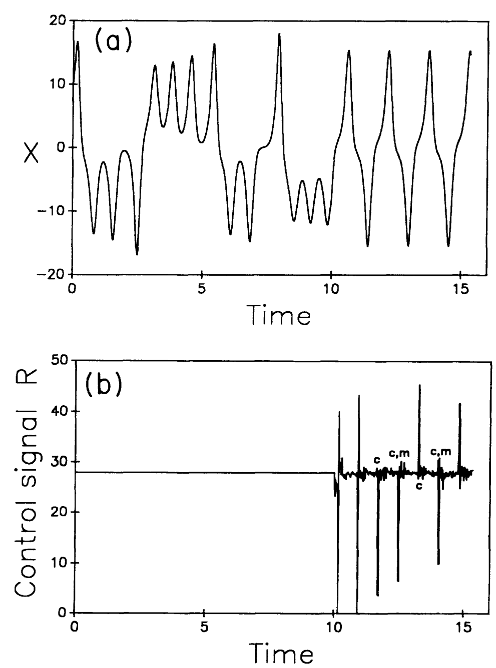

FIG. 4. (a) Velocity of the fluid in the thermosiphon. The control is activated at $t=10$ to track the NPO of Fig. 3. (b) Control actions corresponding to the tracking of the NPO of Fig. 3. (c = crossing of the system trajectory at Σ', m = matching of initial and ending points.)

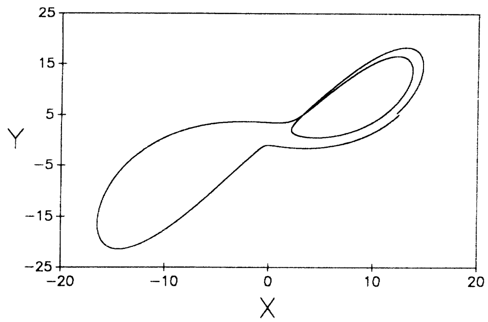

FIG. 5. Nonsymmetric NPO of the Lorenz system (1).

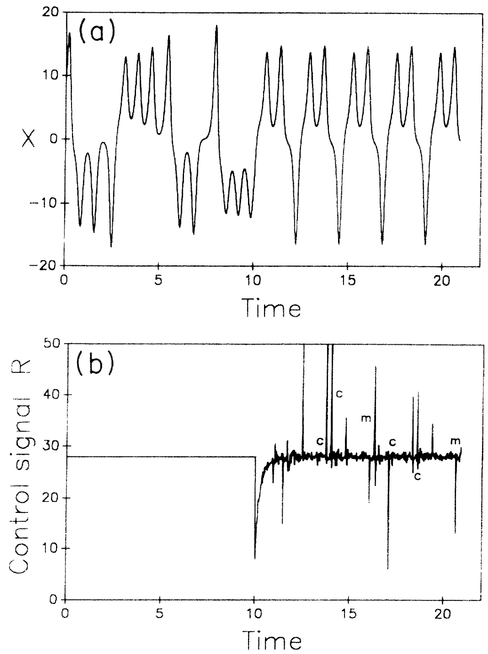

FIG. 6. (a) Velocity of the fluid in the thermosiphon. The control is activated at $t=10$ to track the NPO of Fig. 5. (b) Control actions corresponding to the tracking of the NPO of Fig. 5. (c = crossing of the system trajectory at Σ', m = matching of initial and ending points.)

only for $R=R_{\text{ref}}$. There are two special points where the control signal $R(t)$ suffers bursting: the first one is when the system trajectory crosses the singularity set Σ', and the second one when some control actions $[R(t)\neq 28]$, must be applied in order to match the starting point $\phi_0(x,y,z)$ with the ending point $\phi_T(x,y,z)$. More complex NPO's can be obtained numerically, as shown in Fig. 5. Analogously to the case of the NPO in Fig. 3, Fig. 6(a) shows the velocity in the loop as a time function, and Fig. 6(b) shows the control actions. Also, except at isolated points, the control signal moves in the neighborhood of the stationary value $R_{\text{ref}}=28$.

V. CONCLUSIONS

We have presented a nonlinear feedback control for the Lorenz system. An input-output linearization approach allows us to address efficiently the stabilization of both equilibrium points and periodic orbits. The resulting controller is given by a differential equation, which displays a feedback structure. As a consequence, one has the freedom of choosing an initial condition for the control signal. Finally, this work on the control of the Lorenz equation can be seen as a complement to the classical approach of Hartley and Mossayebi [9].

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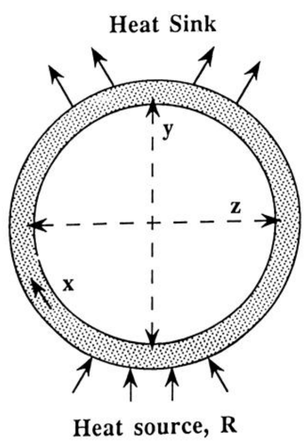

FIG. 1. Schematic description of a thermosiphon.